Workshop on Elastic Stability

Oxford Solid Mechanics

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Elastic Stability, the Energy Criterion, and Bifurcation Theory

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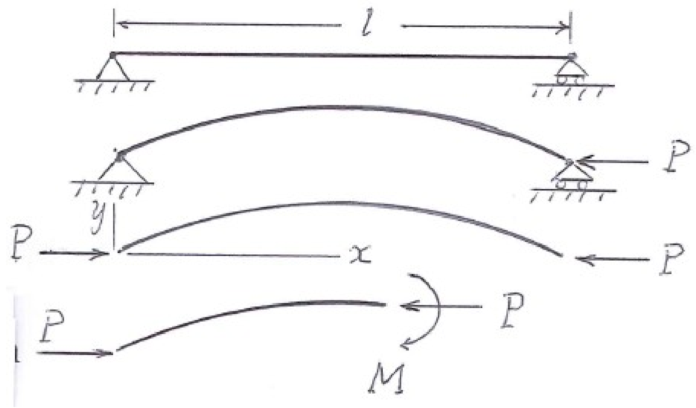
The theory of elastic stability appears to be initiated in Euler's 1744 work on elastic curves. Since then, tremendous progress has been made by many, including Lagrange (1788), Dirichlet (1846), Kirchhoff (1877), Poincaré (1881), Lyapunov (1892), and others.

The scope of elastic stability has been expanded from studying the buckling of elastic rods to a wide spectrum of phenomena. Stability has become a primitive concept valid in very general circumstances.

However, some fundamental questions remain open, and new issues of conceptual importance continue to emerge. Some of them can trace back to Euler's work.

Instability (buckling) of a simply supported rod subjected to a compressive load *P*



Undeformed state

Buckled state

FBD of the rod

FBD of a part of the rod

Equilibrium: $M = Py$

Constitutive relation: $M = -EI\kappa$

Curvature: $\kappa \cong y''$

A boundary value problem

$$\mu y'' + Py = 0$$

$$y(0) = y(l) = 0$$

Flexural rigidity: $\mu = EI$

Solutions of the boundary value problem

If $P \neq \frac{n^2\pi^2\mu}{l^2}, n = 1, 2, \ldots,$ there is only a trivial solution $y(x) = 0,$ which corresponds to the undeformed state.

If $P = \frac{n^2\pi^2\mu}{l^2},$ there are infinitely many solutions

$$y(x) = a \sin \frac{n\pi x}{l}, a \in \mathbb{R},$$

which correspond to the buckled states.

The value of the load at which the buckle first occurs is called critical load

$$P_{cr} = \frac{\pi^2\mu}{l^2}.$$

In a design task, one takes $P \leq \frac{P_{cr}}{f_s}.$

f_s: Factor of safety.

Some apparent discrepancies

Question 1. Why the predicted values of buckling load are discrete

$$P = \frac{\pi^2\mu}{l^2}, \frac{4\pi^2\mu}{l^2}, \frac{9\pi^2\mu}{l^2}, \ldots,$$

while in reality we see a continuous spectrum of the values of buckling load.

Question 2. Why does the theory predict an arbitrary amplitude of the bucking mode $(a \in \mathbb{R})$, while in reality we see a definite amplitude for a given load.

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Question 2. Why does the theory predict an arbitrary amplitude of the bucking mode $(a \in \mathbb{R})$, while in reality we see a definite amplitude for a given load.

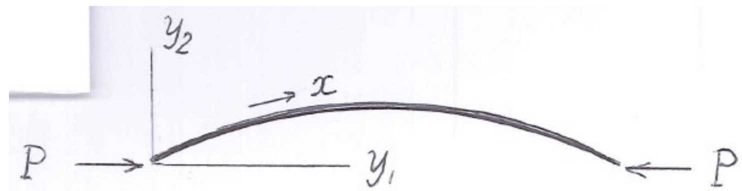
Answer. This theory assumes small deformations

$$\| y \| \ll l,$$

and is invalid for large deformations.

A nonlinear elastic rod theory

The deformed rod is assumed to be a plane curve given by a smooth function $\boldsymbol{y} : [0, l] \to \mathbb{R}^2$.



$\boldsymbol{y}(x)$ is the position vector of a material point that has the coordinate x in the underformed state. The rod is further assumed to be inextensible, i.e., $|\boldsymbol{y}'(x)| = 1.$

Equilibrium:

$$M = Py_2$$

Constitutive relation:

$$M = -\mu\kappa$$

Curvature:

$$\kappa = \boldsymbol{y}' \wedge \boldsymbol{y}''$$

A boundary value problem of nonlinear ODE

$$\mu \boldsymbol{y}' \wedge \boldsymbol{y}'' + P y_2 = 0, \qquad |\boldsymbol{y}'| = 1$$

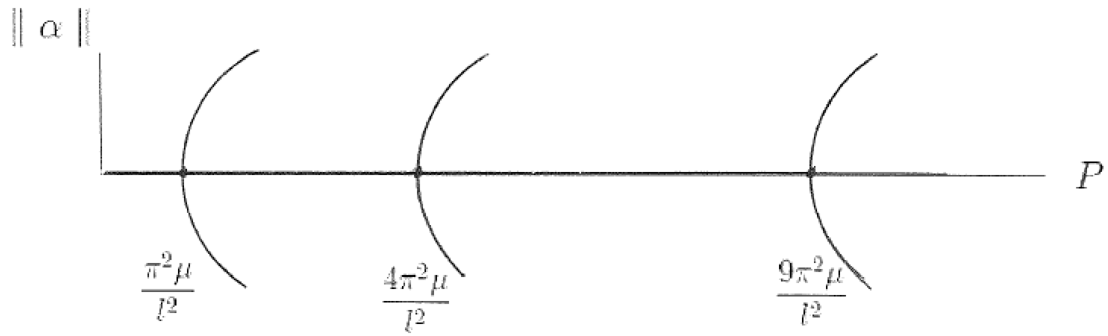
$$\boldsymbol{y}(0) = \boldsymbol{0}, \; y_2(l) = 0$$

A change of variables $\alpha(x) = \sin^{-1} y_2'(x),$ the angle between the tangent of the rod and the y_1 axis.

$$\mu\alpha'' + P\sin\alpha = 0$$

$$\alpha'(0) = \alpha'(l) = 0$$

The solutions can be expressed in terms of the elliptic functions



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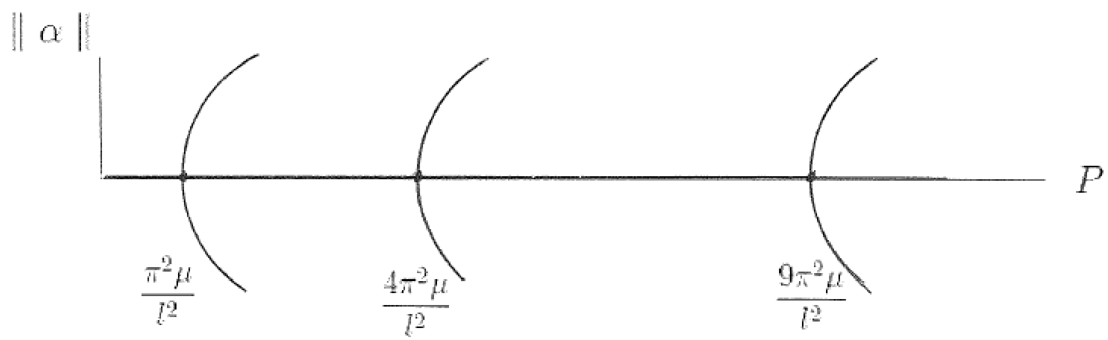

This may seem good. But …

Question 3. When $P > \frac{\pi^2 \mu}{l^2}$, why is the unbuckled state not observed?

Question 4. Why are the higher order buckled states not observed?

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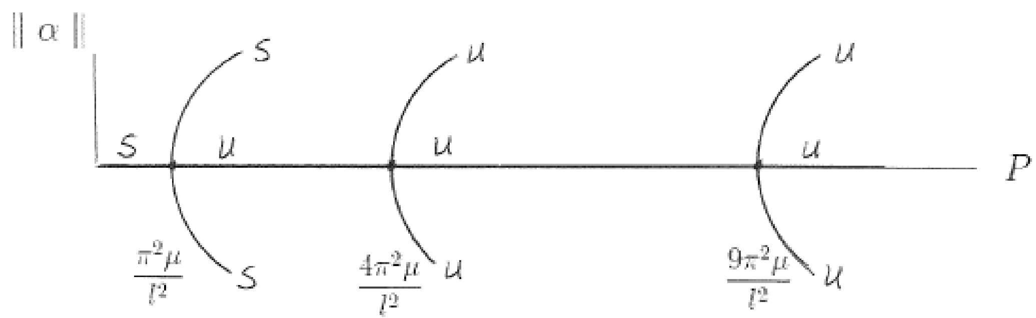

Answer. These states are unstable, while the unbuckled state for $P < \dfrac{\pi^2\mu}{l^2}$ and the first order buckled states are stable.

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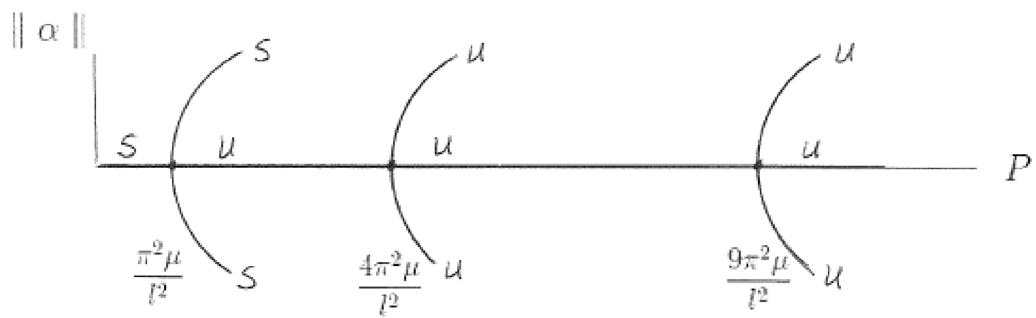

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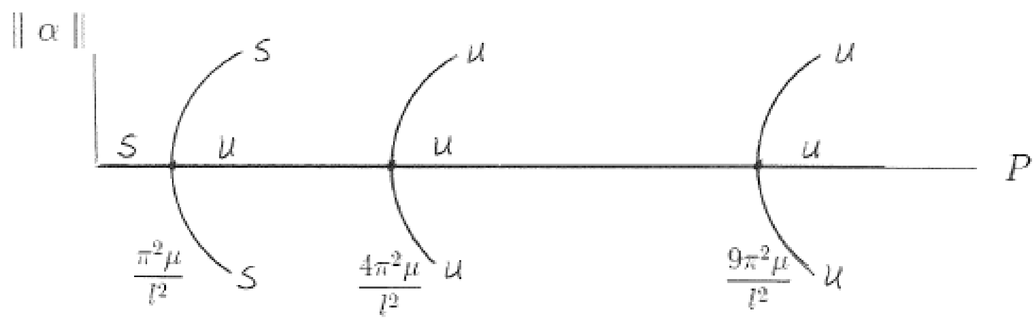

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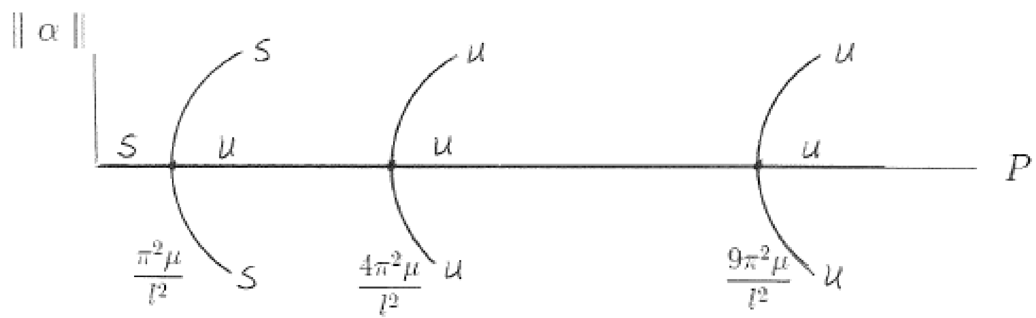

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Question 5. Precise definition of stability/instability? Need a clear statement of stability criteria.

Possible answer 1. Adjacent equilibrium criterion. A primary equilibrium state becomes unstable when there are other equilibrium states nearby.

Possible answer 2. Energy criterion. An equilibrium state is stable if certain potential energy assumes, at the equilibrium state, a minimum in the class of kinematically admissible states.

Stability analysis of the Euler rod by the energy criterion

Potential energy: $I(\boldsymbol{y}) = \int_0^l \frac{1}{2}\mu|\boldsymbol{y}''|^2 dx + P[y_1(l) - l]$

A stable equilibrium solution can be found by minimising $I(\boldsymbol{y})$ in a class of functions that satisfy $|\boldsymbol{y}'| = 1$ in $[0, l]$, and $\boldsymbol{y}(0) = \boldsymbol{0}, y_2(l) = 0.$

The first variation condition: $\delta I = \int_0^l \mu\boldsymbol{y}'' \cdot \delta\boldsymbol{y}'' dx + P\delta y_1(l) = 0$

The Euler-Lagrange equation (equations of equilibrium)

$$\mu\boldsymbol{y}''' + \lambda\boldsymbol{y}' = P\boldsymbol{e}_1, \quad |\boldsymbol{y}'| = 1$$

$$\boldsymbol{y}(0) = \boldsymbol{0}, y_2(l) = 0, \boldsymbol{y}''(0) = \boldsymbol{y}''(l) = \boldsymbol{0}$$

The trivial solution: $\boldsymbol{y}(x) = x\boldsymbol{e}_1, \ \lambda = P$

The second variation condition:

$$\delta^2 I = \int_0^l \mu(|\delta\boldsymbol{y}''|^2 + \boldsymbol{y}'' \cdot \delta^2\boldsymbol{y}'')dx + P\delta y_1^2(l) = \int_0^l (\mu|\delta\boldsymbol{y}''|^2 - P|\delta\boldsymbol{y}'|^2)dx \geq 0$$

At the trivial solution, the last inequality holds if and only if $P \leq \frac{\pi^2\mu}{l^2}$. It can also be shown that the second variation condition is not satisfied for the higher order buckled solutions.

Question 6. Why does the energy stability criterion work? Can it be justified by more fundamental physical principles? Can it be applied to other problems (3-dimensional nonlinear elasticity, thermoelasticity, viscoelasticity, fluid mechanics, ...)?

Re-examination of the concept of stability

Ordinary dictionary meaning: Steadiness, firmness.

Latin word *stabilitate*: Immovability.

Foscarini (1699) "De mobilitate terrae et stabilitate solis" (On motion of Earth and immovability of the Sun).

Definition of elastic stability from dynamics, and a justification of the energy criterion.

Lagrange (1788) proposed that an equilibrium of a discrete dynamic system is stable if, in displacing the points of the system by an infinitesimal amount, the displacements remain, throughout the course of the motion, contained between certain limits. He showed, by linearising the equations of motion about the equilibrium, that if the potential energy assumes a minimum at the equilibrium, then this position is stable.

Dirichlet (1846) criticised Lagrange's proof for neglecting higher order terms, and provided a correct proof by describing bounds to the perturbations of the initial data and the subsequent motion.

The Lagrange-Dirichlet criterion is commonly called the "energy test", or "energy stability criterion". It has been further developed to the more general "Lyapunov direct method", and has enjoyed a widespread applications.

The importance of such a method centres round the fact that the dynamic stability of a solution may be ascertained without the explicit knowledge of that and other dynamic solutions.

Dynamics of the Euler rod

Motion of the rod $\boldsymbol{y} \in C^2([0,l] \times [0,\infty); \mathbb{R}^2)$

Equations of motion

$$\rho \boldsymbol{y}_{tt}(x,t) + \boldsymbol{f}_x(x,t) = 0,\ M_x(x,t) + \boldsymbol{y}_x(x,t) \wedge \boldsymbol{f}(x,t) = 0$$

Inextensibility $|\boldsymbol{y}_x(x,t)| = 1$

Constitutive relation $M(x,t) = \mu \boldsymbol{y}_{xx}(x,t) \wedge \boldsymbol{y}_x(x,t)$

Boundary conditions

$$\boldsymbol{y}(0,t) = \boldsymbol{0},\ y_2(l,t) = 0,\ \boldsymbol{f}(l,t) = P\boldsymbol{e}_1,\ M(0,t) = M(l,t) = 0$$

Initial conditions

$$\boldsymbol{y}(x,0) = \boldsymbol{y}_0(x),\ \boldsymbol{y}_t(x,0) = \boldsymbol{0}$$

For a time-independent function $\boldsymbol{y}(x,t) = \boldsymbol{y}_0(x),$ this initial-boundary value problem of dynamics reduces to the boundary value problem of equilibrium.

Work-energy principle

Taking the inner product of $\boldsymbol{y}_t$ and the equation of motion, integrating over [0, l], and using the boundary conditions, we find that

$$\frac{d}{dt}\left[\int_0^l \left(\frac{1}{2}\rho|\boldsymbol{y}_t(x,t)|^2 + \frac{1}{2}\mu|\boldsymbol{y}_{xx}(x,t)|^2\right)dx + Py_1(l,t)\right] = 0$$

Define a Lyapunov function

$$V(t) = \int_0^l \left(\frac{1}{2}\rho|\boldsymbol{y}_t(x,t)|^2 + \frac{1}{2}\mu|\boldsymbol{y}_{xx}(x,t)|^2\right)dx + P[y_1(l,t) - l]$$

$$= \int_0^l \frac{1}{2}\rho|\boldsymbol{y}_t(x,t)|^2 dx + I(\boldsymbol{y}(x,t))$$

where

$$I(\boldsymbol{y}(x,t)) = \int_0^l \frac{1}{2}\mu|\boldsymbol{y}_{xx}(x,t)|^2 dx + P[y_1(l,t) - l].$$

The Lyapunov function is constant along a dynamic solution. By using the initial conditions, we have

$$\int_0^l \frac{1}{2}\rho|\boldsymbol{y}_t(x,t)|^2 dx + I(\boldsymbol{y}(x,t)) = I(\boldsymbol{y}_0(x)).$$

This implies that $$I(\boldsymbol{y}(x,t)) \le I(\boldsymbol{y}_0(x)).$$

The minimum energy stability criterion and the dynamic stability criterion

Suppose that the potential energy assumes a strict local minimum at an equilibrium state $\bar{\boldsymbol{y}}(x)$ of the Euler rod, that is,

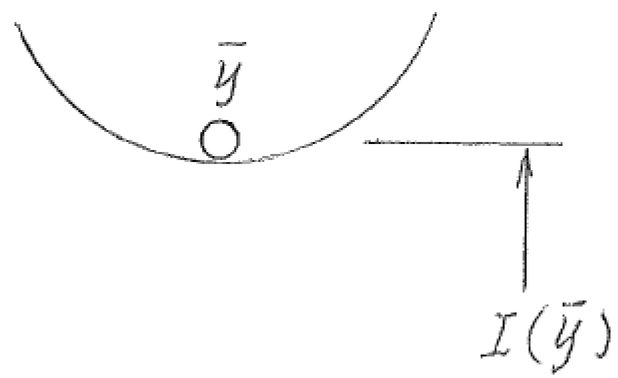
$$I(\bar{\boldsymbol{y}}) < I(\boldsymbol{y}) \quad \forall\, \boldsymbol{y} \in \mathcal{N} - \{\bar{\boldsymbol{y}}\},$$

where $\mathcal{N}$ is a properly chosen neighbourhood of $\bar{\boldsymbol{y}}(x)$. Perturb the rod slightly to a new state $\boldsymbol{y}_0(x)$, which is in $\mathcal{N}$, but may not be in equilibrium. The ensuing dynamic s $\boldsymbol{y}(x,t)$ with $\boldsymbol{y}_0(x)$ being the initial position must satisfy

$$I(\bar{\boldsymbol{y}}(x) \leq I(\boldsymbol{y}(x,t)) \leq I(\boldsymbol{y}_0(x)) \quad \forall\, t > 0.$$

That is, the potential energy associated with the dynamic solution $\boldsymbol{y}(x,t)$ is bounded between those of the equilibrium solution $\bar{\boldsymbol{y}}(x)$ and of the perturbed state $\boldsymbol{y}_0(x)$. If we can find a metric under which the above boundedness implies that the dynamic solution is "close" to the equilibrium solution, we would have a justification of the minimum energy stability criterion from the dynamic stability criterion.

A dynamic analogy



An equilibrium state $\bar{y}$ with potential energy $I(\bar{y})$

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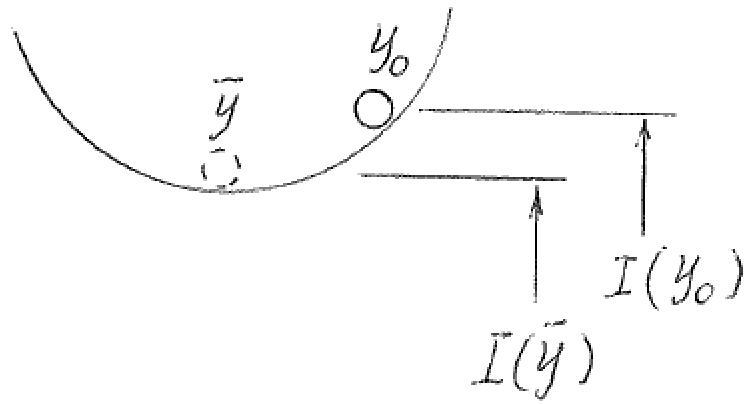

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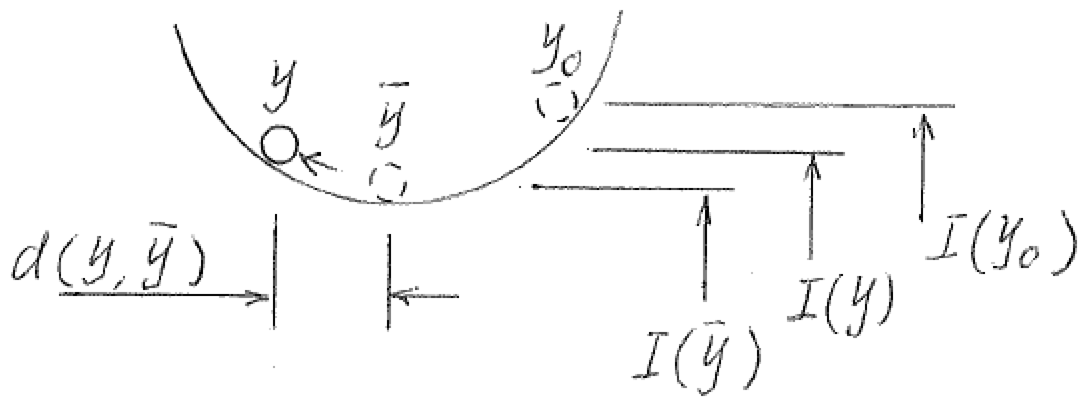

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The ensuing motion y with potential energy $I(y)$.

$I(\bar{y}) \leq I(y) \leq I(y_0)$ $\Rightarrow$ $d(y,\bar{y})$ small?

A possible metric for closeness

The set of kinematically admissible states

$$\mathcal{Y} = \{\boldsymbol{y} \in C^2([0,l];\mathbb{R}^2) : |\boldsymbol{y}'| = 1, \boldsymbol{y}(0) = \boldsymbol{0}, y_2(l) = 0\}$$

A technical difficulty lies in the fact that $\mathcal{Y}$ is not a linear space due to the inextensibility constraint, and the usual C^2 norm alone may not represent a good measure of "closeness" for $\mathcal{Y}$. A possible measure is given by the metric

$$d(\boldsymbol{f},\boldsymbol{g}) = \int_0^l \left[\frac{2\pi^2}{l^2}(\boldsymbol{f}'\cdot\boldsymbol{g}' - 1) + |(\boldsymbol{f}-\boldsymbol{g})''|^2\right]dx$$

We wish to examine the stability of the trivial solution $\bar{\boldsymbol{y}}(x) = x\boldsymbol{e}_1$. We have $I(\bar{\boldsymbol{y}}) = 0$, and

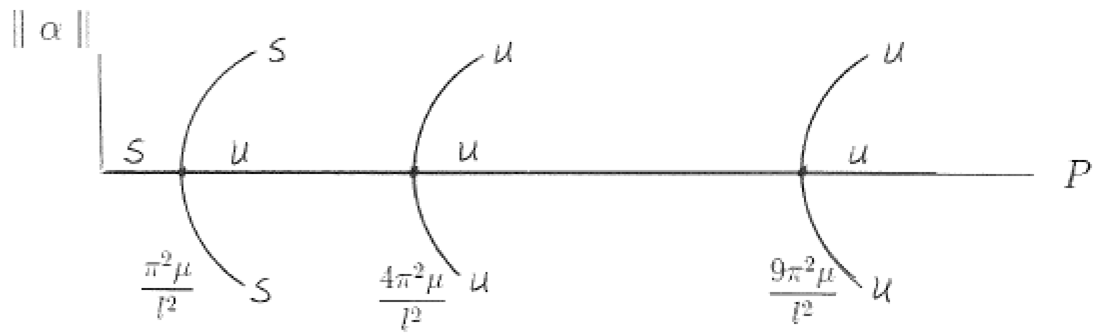
$$I(\boldsymbol{y}) = \frac{l^2}{2\pi^2}\left[P\, d(\boldsymbol{y},\bar{\boldsymbol{y}}) + \left(\frac{\pi^2\mu}{l^2} - P\right)\int_0^l |\boldsymbol{y}''|^2 dx\right]$$

When $P < \frac{\pi^2\mu}{l^2}$, the potential assumes a strict local minimum at the trivial solution Furthermore, the bounds on the potential energy associated with the dynamic solution implies that

$$0 \le P\, d(\boldsymbol{y},\bar{\boldsymbol{y}}) + \left(\frac{\pi^2\mu}{l^2} - P\right)\int_0^l |\boldsymbol{y}''|^2 dx \le P\, d(\boldsymbol{y}_0,\bar{\boldsymbol{y}}) + \left(\frac{\pi^2\mu}{l^2} - P\right)\int_0^l |\boldsymbol{y}_0''|^2 dx$$

This may lead to the conclusion that the dynamic solution $\boldsymbol{y}(x,t)$ stays close to the equilibrium solution, and hence justify the energy stability criterion from dynamics.

Relation between stability and bifurcation



The set of the equilibrium solutions exhibits the phenomenon of bifurcation, i.e., the number of solutions undergoing a change as the parameter *P* passes through certain points, called bifurcation points. It is observed that at the first bifurcation point, the trivial solution (unbuckled state) loses stability, giving rise to stable non-trivial solutions (buckled states). Is this always true?

Ericksen & Toupin (1956), Hill (1957): If an equilibrium solution is stable in the sense that the second variation of the potential energy is positive definite, there is no bifurcation.